Solutions of partial differential equations:

Any relation between the dependent variable and the independent variables containing the same number of arbitrary constants as the number of independent variables which satisfies a pde involving these variables (dependent & independent both) is called a complete integral of the pde. For example, suppose we have the dependent variable z as a function of x and y;

$z = z(x, y)$, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

and the pde is $f(x, y, z, p, q) = 0 \rightarrow ①$

Then a function $F(x, y, z, a, b) = 0$, where a and b are arbitrary constants, $\rightarrow ②$ satisfying ① is a complete integral of ①.

If we give some particular values to a & b w.r.t. some given conditions, then we have a particular integral. If $b = \phi(a)$, then it is called general integral.

Envelope of the surface given by ② is called the singular integral of ①. It is obtained by eliminating a & b from

$$F(x, y, z, a, b) = 0, \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0.$$

The method of Lagrange subsidiary equations for the Lagrange linear equations $Pp + Qq = R$ given by $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ has already been discussed.

Some special type of equations:

Type I: Equations involving only p & q:

$$f(p, q) = 0$$

Complete integral is $z = ax + by + c$

where $f(a, b) = 0$, a and c are arbitrary constants. $\rightarrow ③$

As $\frac{\partial z}{\partial x} = p = a$ & $\frac{\partial z}{\partial y} = q = b$.

From ③ if $b = \phi(a)$, then

$z = ax + \phi(a)y + c \longrightarrow (4)$

if $c = \psi(a)$, then $z = ax + \phi(a)y + \psi(a) \longrightarrow (5)$

Diff (5) w.r.t. a, $0 = x + \phi'(a)y + \psi'(a) \longrightarrow (6)$

Eliminating a between (5) & (6) gives general integral.

For singular integral, differentiate (4) partially w.r.t. a & c

$$0 = x + \phi'(a)y$$
$$0 = 1$$

So there is no singular integral.

Ex $(x^2+y^2)(p^2+q^2) = 1. \longrightarrow (7)$

Put $x = r\cos\theta$ and $y = r\sin\theta$.

$\Rightarrow r^2 = x^2+y^2$ and $\tan\theta = \frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{-y}{r^2} = -\frac{\sin\theta}{r} \qquad \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial\theta}d\theta = \cancel{P\,dr + Q\,dy}\; P\,dr + Q\,d\theta$$

where $P = \frac{\partial z}{\partial r}$, $Q = \frac{\partial z}{\partial\theta}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial z}{\partial\theta}$$
$$= P\cos\theta - Q\frac{\sin\theta}{r}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial y}$$
$$= P\sin\theta + Q\frac{\cos\theta}{r}$$

$$(7) \Rightarrow r^2\Big(P^2\cos^2\theta + \frac{Q^2}{r^2}\sin^2\theta - 2\frac{PQ}{r}\cos\theta\sin\theta + P^2\sin^2\theta + \frac{Q^2}{r^2}\cos^2\theta + 2\frac{PQ}{r}\cos\theta\sin\theta\Big) = 1$$

$$\Rightarrow r^2\left(P^2 + \frac{Q^2}{r^2}\right) = 1 \quad\text{or}\quad r^2P^2 + Q^2 = 1$$

Put $r = e^u \Rightarrow \frac{dr}{du} = r$

So $P = \frac{\partial z}{\partial r} = \frac{\partial z}{\partial u}\frac{du}{dr} = \frac{1}{r}\frac{\partial z}{\partial u}$

$\Rightarrow P_1 = \frac{\partial z}{\partial u} = rP \Rightarrow P_1^2 + Q^2 = 1$

$$z = au + b\theta + c \quad \text{where} \quad a^2+b^2=1 \Big\} \text{ complete integral.}$$

$$\Rightarrow z = au + \sqrt{1-a^2}\,\theta + c$$

$$\cancel{= a \log r} \quad = a\ln r + \sqrt{1-a^2}\tan^{-1}\frac{y}{x} + c$$

$$= \frac{a}{2}\ln(x^2+y^2) + \sqrt{1-a^2}\tan^{-1}\frac{y}{x} + c.$$

$c = \psi(a) \quad \Rightarrow$

$$z = \frac{a}{2}\ln(x^2+y^2) + \sqrt{1-a^2}\tan^{-1}\frac{y}{x} + \psi(a)$$

Diff w.r.t. a

$$0 = \frac{1}{2}\ln(x^2+y^2) + \frac{1}{\sqrt{1-a^2}}(-a)\tan^{-1}\frac{y}{x} + \psi'(a)$$

Eliminating a gives general integral.

Type II: Equations having p, q and z.

$$f(p,q,z)=0$$

Trial solution: $z = f(x+ay)$, a arbitrary

$X = x+ay$

$z = f(X)$

$$\cancel{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX}\frac{\partial X}{\partial x} = \frac{dz}{dX}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX}\frac{\partial X}{\partial y} = a\frac{dz}{dX}$$

$$\Rightarrow f\left(\frac{dz}{dX}, a\frac{dz}{dX}, z\right) = 0 \qquad \text{Integrate this.}$$

EX $\quad q^2y^2 = z(z-px)$.

Put $x = e^X$ and $y = e^Y$

$$\frac{dx}{dX} = e^X = x \qquad \frac{dy}{dY} = e^Y = y$$

$$dz = \frac{\partial z}{\partial X}dX + \frac{\partial z}{\partial Y}dY = P\,dX + Q\,dY \qquad P = \frac{\partial z}{\partial X}, \quad Q = \frac{\partial z}{\partial Y}$$

$$p = \frac{\partial z}{\partial x} = P\frac{1}{x} \Rightarrow P = xp$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{1}{y} \Rightarrow Q = yq$$

$$\Rightarrow Q^2 = z(z-P) \quad \Rightarrow \quad z^2 - zP = Q^2$$

$$U = x + ay$$

$$P = \frac{\partial z}{\partial x} = \frac{dz}{dU}$$

$$Q = \frac{\partial z}{\partial y} = a\frac{dz}{dU}$$

$$z^2 - z\frac{dz}{du} = a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{dU} - z^2 = 0$$

$$\Rightarrow \frac{dz}{dU} = \frac{-z \pm \sqrt{z^2 + 4a^2z^2}}{2a^2} = \lambda z \quad \text{wher}$$

$$\lambda = \frac{-1 \pm \sqrt{1+4a^2}}{2a^2}$$

$$\Rightarrow \frac{dz}{z} = \lambda dU$$

$$\Rightarrow \frac{1}{\lambda}\ln z = \cancel{\lambda U + b}\; U + \ln b.$$

$$\Rightarrow \frac{1}{\lambda}\ln z = x + ay + \ln b$$

$$\frac{1}{\lambda}\ln z = \ln x + a\ln y + \ln b$$

$$\ln z^{1/\lambda} = \ln xy^a b$$

$$\Rightarrow z^{1/\lambda} = xy^a b \quad , \quad a, b \text{ arbitrary.}$$

Type III: Equations of the type $f(x,p) = F(y,q)$.

Put $f(x,p) = F(y,q) = a$ → (1) → arbitrary constant.

This is a trial solution.

(1) $\Rightarrow$ $p = f_1(x,a)$ & $q = f_2(y,a)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$$

$$= f_1(x,a)\,dx + f_2(y,a)\,dy$$

$$\Rightarrow z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$$

where b is arbitrary constant.

Ex $z^2(p^2+q^2) = x^2+y^2$

Put $U = \frac{z^2}{2}$ $\Rightarrow$ $\frac{dU}{dz} = z$

$P = \frac{\partial U}{\partial x} = \frac{dU}{dz}\frac{\partial z}{\partial x} = zp$ & $Q = \frac{\partial U}{\partial y} = \frac{dU}{dz}\frac{\partial z}{\partial y} = zq$

$\Rightarrow$ $P^2 + Q^2 = x^2 + y^2$

$\Rightarrow$ $-x^2 + P^2 = y^2 - Q^2 = a$

So $P = \sqrt{a+x^2}$ and $Q = \sqrt{y^2-a}$

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy.$$

$$= P\,dx + Q\,dy$$

$$= \sqrt{a+x^2}\,dx + \sqrt{y^2-a}\,dy$$

$$\Rightarrow \frac{z^2}{2} = U = \int\sqrt{a+x^2}\,dx + \int\sqrt{y^2-a}\,dy$$

$$= \frac{x}{2}\sqrt{a+x^2} + \frac{a}{2}\ln\left(x+\sqrt{a+x^2}\right)$$

$$+ \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\ln\left(y+\sqrt{y^2-a}\right) + b/2$$

$$\Rightarrow z^2 = x\sqrt{a+x^2} + y\sqrt{y^2-a} - a\ln\left(\frac{x+\sqrt{a+x^2}}{y+\sqrt{y^2-a}}\right) + b.$$

Scanned with CamScanner

Type IV: Equations of the Type $z = px + qy + f(p, q)$

This is analogous to Clairaut's Form.

The complete integral is

$$z = ax + by + f(a, b)$$

For general integral, put $b = \varphi(a)$ and eliminate a from

$$F = z - ax - by - f(a, \varphi(a)) = 0 \quad \& \quad \frac{\partial F}{\partial a} = 0$$

For singular integral eliminate a & b from

$$F = z - ax - by - f(a, b) = 0, \quad \frac{\partial F}{\partial a} = -x - \frac{\partial f}{\partial a} = 0$$

$$\& \quad \frac{\partial F}{\partial b} = -y - \frac{\partial f}{\partial b} = 0.$$

Ex $\quad z = px + qy - 2\sqrt{pq}$

complete integral is $\quad z = ax + by - 2\sqrt{ab}$

$\Rightarrow F = z - ax - by + 2\sqrt{ab} = 0$

for singular solution

$$\frac{\partial F}{\partial a} = -x + \frac{2b}{2\sqrt{ab}} = 0 \quad \Rightarrow \quad \sqrt{b} - \sqrt{a}\, x = 0$$

$$\frac{\partial F}{\partial y} = -y + \frac{2a}{2\sqrt{ab}} = 0 \quad \Rightarrow \quad \sqrt{a} - \sqrt{b}\, y = 0$$

So $\quad \sqrt{a}\, x = \sqrt{b} \qquad \qquad \sqrt{b}\, y = \sqrt{a}$

$\Rightarrow ax = \sqrt{ab} \qquad \qquad \Rightarrow by = \sqrt{ab}$

$$\Rightarrow F = z - \sqrt{ab} - \sqrt{ab} + 2\sqrt{ab} = 0$$

$\&$ $\quad \sqrt{b}\, xy = \sqrt{b} \quad \Rightarrow xy = 1$ This is the singular integral.

Ex $z = px + qy + \ln pq$

Complete integral $z = ax + by + \ln ab$

$F = z - ax - by - \ln ab = 0$

$\frac{\partial F}{\partial a} = -x - \frac{1}{ab} b = 0 \Rightarrow a = -1/x$

$\frac{\partial F}{\partial b} = -y - \frac{1}{ab} a = 0 \Rightarrow b = -1/y$

$\Rightarrow F = z + 1 + 1 - \ln \frac{1}{xy} = 0$

$\Rightarrow z + 2 + \ln xy = 0 \rightarrow$ Singular integral